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NON-TREEABILITY FOR PRODUCT GROUP ACTIONS*

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ABSTRACT

A product of two lcsc non-compact groups $G_1 \times G_2$ acting freely by measure preserving transformation on a standard Borel probability space gives rise to a non-treeable equivalence relation unless both groups are amenable.

The result of the present paper can be placed as one of a series.

Theorem 0.1 (Adams, [1]): *Let E_1, E_2 be countably infinite measure preserving equivalence relations on standard Borel probability spaces X_1, X_2 with $E_1 \times E_2$ non-amenable. Then $E_1 \times E_2$ is non-treeable.*

Theorem 0.2 (Gaboriau, [11]): *Let G_1, G_2 be countably infinite groups with $G_1 \times G_2$ non-amenable. Then any free measure preserving action of $G_1 \times G_2$ on a standard Borel probability space is non-treeable.*

Theorem 0.3 (Kechris, [18]): *Let G_1, G_2 be locally compact non-compact second countable groups one of which contains a non-compact closed amenable subgroup and with $G_1 \times G_2$ non-amenable. Then any free measure preserving action of $G_1 \times G_2$ on a standard Borel probability space is non-treeable.*

Theorem 0.4 (Pemantle, Peres, [26]): *Let X and Y be infinite, locally finite graphs. Suppose that $G \subset \mathrm{Aut}(X)$ is a closed nonamenable subgroup of*

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$\mathrm{Aut}(X)$, and $H \subset \mathrm{Aut}(Y)$ has an infinite orbit. Then there is no $G \times H$-invariant probability measure on the set of spanning trees of the direct product graph $G \times H$.

While here we show:

THEOREM 0.5: *Let G_1, G_2 be locally compact second countable groups with $G_1 \times G_2$ acting in a measure preserving, measurable manner on a standard Borel probability space with the orbit equivalence relation E_{G_1} non-amenable and G_2 not acting essentially transitively on any of its ergodic components. Then the orbit equivalence relation induced by the product group action, $E_{G_1 \times G_2}$, is not treeable.*

COROLLARY 0.6: *Let G_1, G_2 be locally compact non-compact second countable groups with $G_1 \times G_2$ acting in a free, measure preserving, measurable manner on a standard Borel probability space. Then if $G_1 \times G_2$ is non-amenable then $E_{G_1 \times G_2}$ is not treeable.*

Our proof is a refinement of Kechris', which, in turn, had points in common with Adams', and as thus traces back to ideas from Zimmer, [31], and from there to earlier work of Margulis and Mostow. The original proof of Gaboriau's result, however, used a radically different idea to these three results, and the proof of Pemantle and Peres was different again. It is clear that 0.5 implies 0.3 and 0.2; in the last section we use some observations from [20] to show that 0.4 follows as well, but this is less of an achievement since their proof is by far the simplest and most elegant of the sequence. Again as pointed out by [20], 0.4 can be used to derive 0.2, though it should also be said that 0.2 arises as one small piece of a much broader body of work on the concept of *cost*. 0.5 also implies 0.1, since any countable measure preserving equivalence relation is induced by the measure preserving action of a countable group ([9]).

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final statement of 0.5 is sharper than the original version and borrows directly from a suggestion of the referee.

1. Borel reducibility

In this paper we consider only actions by groups which are locally compact, second countable, and Hausdorff – **lcsc** groups for short. Any such group is completely metrizable (see, for example, [24, Chapter 3, exercise 3]). Thus, in particular, it is Polish, and there is a robust structure theorem for its Borel sets and Borel actions (see for instance [5], [16]).

We will find it convenient to organize the later arguments around the notion of Borel reducibility.

Definition: Let E, F be Borel equivalence relations on standard Borel spaces X, Y. We say that E **is Borel reducible to** F, written $E \leq_B F$, if there is a Borel function

$$\theta : X \to Y,$$

such that for all $x_1, x_2 \in X$

$$x_1 E x_2 \Leftrightarrow \theta(x_1) F \theta(x_2).$$

We say that E is **smooth** if it is Borel reducible to $\mathrm{id}(\mathbb{R})$, the identity relation on $\mathbb{R}$.

Definition: An equivalence relation E on standard Borel X is **treeable** if there is a Borel subset $\mathcal{G} \subset X \times X$ such that:

(i) $\mathcal{G}$ is symmetric;
(ii) $\mathcal{G}$ is irreflexive, in the sense that (x, x) is never in $\mathcal{G}$;
(iii) $\mathcal{G}$ is acyclic (in other words, if $x_1, x_2, \ldots, x_n \in \mathcal{G}$ with each $(x_i, x_{i+1}) \in \mathcal{G}$ and $x_i \neq x_j$ for $i \neq j$, then $(x_n, x_1) \notin \mathcal{G}$);
(iv) the connected components of $\mathcal{G}$ form the equivalence classes of E.

An equivalence relation is **countable** if every equivalence relation is countable.

It has been customary to only consider treeability in the context of countable Borel equivalence relations, but it makes sense to consider treeable equivalence relations with uncountable classes and certain key aspects of the existing theory simply adapts.

LEMMA 1.1 (essentially [14]): *Let E on X be a treeable Borel equivalence relation and let $A \subset X$ be Borel set meeting each orbit in at most a countable set. Then $E|_A$, the restriction of E to A, is again treeable.*

Proof. Let B be the set of points which are either in A or lie between two distinct points in A. By Novikov–Lusin (see [16]), this set is Borel, and it is clear that $E|_B$ is treeable. Now [14] shows that the restriction of a countable treeable equivalence relation to a Borel subset is again treeable. ∎

COROLLARY 1.2: *Let G be lcsc group acting in a Borel manner on a standard Borel space X. Then E_G^X is treeable if and only if E_G^X is Borel reducible to an orbit equivalence relation induced by a free, Borel action of $\mathbb{F}_2$, the free group on two generators.*

Proof. Following [17], we can fix Borel set $A \subset X$ meeting each orbit in a countable non-empty set. By the uniformization theorem for Borel sets in the plane with countable sections[1] we can in a Borel manner select to each $x \in X$ some $x' = \phi(x) \in [x]_G \cap A$, and hence obtain $E_G^X \leq_B E_G^X|_A$.

($\Rightarrow$:) It suffices to show that $E_G^X|_A$ is Borel reducible to a suitable action of $\mathbb{F}_2$. But by the last lemma $E_G^X|_A$ is again treeable, and then since it is countable we have that it is Borel reducible to a free, Borel action of $\mathbb{F}_2$ by [14].

($\Leftarrow$:) Assume $\theta : X \to Y$ gives a reduction of E_G^X to $E_{\mathbb{F}_2}^Y$, where Y is a free, Borel $\mathbb{F}_2$ space. Then in particular $\theta|_A$ reduces $E_G^X|_A$ to a treeable equivalence relation, and hence, again by [14], $E_G^X|_A$ admits some treeing, $\mathcal{G}_A$. We then define a treeing by $x\mathcal{G}x'$ if and only if either:

(i) $x, x' \in A$, $x\mathcal{G}_A x'$; or
(ii) $x \in A$, $x' \notin A$, $\phi(x') = x$; or
(iii) $x \in A$, $x' \notin A$, $\phi(x) = x'$. ∎

The treeable relations are most naturally contrasted with the hyperfinite.

Definition: An equivalence relation E is **hyperfinite** if there is an increasing union of Borel equivalence relations, (F_n), with finite classes, such that

$$E = \bigcup_{n \in \mathbb{N}} F_n.$$

[1] Again, [16] is an ideal place to find an account of classical facts such as this one.

Hyperfinite equivalence relations can be alternately characterized as the equivalence relations induced by a Borel action of $\mathbb{Z}$ or the Borel equivalence relations for which every infinite equivalence class can be assigned, in a uniformly Borel fashion, the structure of a $\mathbb{Z}$-chain.[2] This last characterization makes it clear that hyperfiniteness is a special kind of treeability.

However, whereas most of the structural properties of hyperfinitess are understood, at least up to discounting null sets, the structural properties of the treeable equivalence relations have not been adequately documented.

Question: Is a countable Borel equivalence relation with finite index over treeable again treeable?

See here [14] for a discussion of this problem and terminology. In that paper they show that finite index over *hyperfinite* is again hyperfinite.

Question: Which lcsc groups have all their free Borel, measure preserving actions on standard Borel probability spaces treeable?

In the case of hyperfinite this is largely understood: If every such action of lcsc G is Borel reducible to a hyperfinite equivalence relation, then G is amenable (see e.g. [31]), and conversely if G is amenable then after possibly discounting a null set any such action will be Borel reducible to a hyperfinite equivalence relation (see [31], [25]). (It remains notoriously open, however, even for discrete groups, whether any such action will be hyperfinite everywhere, and not just almost everywhere.)

On the other hand, there is at least one structural property known true for hyperfinite and known false for treeable: Increasing unions of hyperfinite equivalence relations are hyperfinite almost everywhere by [7]; an increasing union of countable treeable equivalence relations need not be treeable on any conull set (see [14] or [11]).

An especially important example of a hyperfinite equivalence relation is given by E_0.

Notation: We let E_0 be the equivalence relation of eventual agreement on infinite binary sequences. So for $\vec{x}, \vec{y} \in \{0,1\}^{\mathbb{N}} (=_{\mathrm{df}} 2^{\mathbb{N}})$ we let

$$\vec{x} E_0 \vec{y}$$

[2] See [22] for an extended analysis of some of the subtle distinctions which are possible here.

if $\exists N\ \forall m > N(x_m = y_m)$.

E_0 is clearly hyperfinite; conversely [8] shows that every hyperfinite equivalence relation is Borel reducible to E_0.

Notation: We let E_1 be the equivalence relation of eventual agreement on infinite sequences of reals: For $\vec{x}, \vec{y} \in \mathbb{R}^{\mathbb{N}}$ we let

$$\vec{x} E_1 \vec{y}$$

if $\exists N\ \forall m > N(x_m = y_m)$.

E_1 is an example of a treeable equivalence relation which is not Borel reducible to any countable Borel equivalence relation. In fact by [19], not Borel reducible to any Borel action of a Polish group.

Notation: We let $E_{\infty\mathcal{T}}$ arise from the orbit equivalence relation of $\mathbb{F}_2$ on $F(2^{\mathbb{F}_2})$, the part of $2^{\mathbb{F}_2}(=_{\rm df}\ \{0,1\}^{\mathbb{F}_2})$ on which every non-trivial element of $\mathbb{F}_2$ has no fixed points under the shift action.

For our purposes the exact definition of $E_{\infty\mathcal{T}}$ is unimportant. We only need the fact established in [14] that it is induced by a free action of $\mathbb{F}_2$ and that any countable, Borel, treeable equivalence relation is Borel reducible to $E_{\infty\mathcal{T}}$. With this notation granted, I can finish this brief survey with one last, and, for the author, most central question:

Question: Up to $\leq_B$-reducibility, how many Borel equivalence relations E are there with

$$E_0 <_B E < E_{\infty\mathcal{T}}?$$

2. Mixing properties

Definition: A locally compact group G equipped with a Haar measure ν is said to act **measurably** on a standard Borel probability space (X, μ) if for any Borel $A \subset X$ the set $\{(g, x) : g \cdot x \in A\}$ is measurable in $(G \times X, \nu \times \mu)$. Let G be a lcsc group acting in a properly ergodic, measurable, non-singular manner on a standard Borel probability space (X, μ). The action of G on (X, μ) is **mildly mixing** if whenever G acts in a properly ergodic, measurable, non-singular, manner on a standard Borel probability space (X', μ'), the diagonal action of G on $(X \times X', \mu \times \mu')$ is again ergodic. ([28])

For $\mathcal{M}$ a class of separable Banach spaces, the action of G on (X, μ) is **$\mathcal{M}$-ergodic** if whenever $E \in \mathcal{M}$ and we have a continuous homomorphism

$$\pi : G \to \mathrm{Iso}(E),$$

$$g \mapsto \pi_g,$$

from G to the isometry group of E, and we have a measurable

$$\varphi : X \to E$$

which is equivariant in the sense that

$$\varphi(g \cdot x) = \pi_g(\varphi(x))$$

all $x \in X, g \in G$, then φ is constant almost everywhere.

A couple of remarks about these definitions.

There is essentially only one standard Borel structure on a separable Banach space, and that is the Borel structure arising from the Polish topology induced by the Banach norm. This is the sense in which we require $\varphi : X \to E$ to be measurable.

Since the isometry group of separable Banach space is Polish, it follows from Pettis' lemma, [27], that any measurable homomorphism $G \to \mathrm{Isom}(E)$ is necessarily continuous.

Future discussions of $\mathcal{M}$-ergodicity will probably be clarified by noting that a standard argument shows that an almost every equivariant map from X to E can be replaced by an everywhere equivariant map.

Lemma 2.1: *Let G be a locally compact group equipped with right invariant Haar measure acting in a non-singular measurable manner on standard Borel probability space (X, μ). Let $\pi : G \to \mathrm{Isom}(E)$ be a continuous homomorphism and $\hat{\varphi} : X \to E$ a measurable map with almost every $x \in X$, $g \in G$*

$$\hat{\varphi}(g \cdot x) = \pi_g(\hat{\varphi}(x)).$$

Then we can find conull, invariant $X_0 \subset X$ and $\varphi : X_0 \to E$ agreeing almost every with $\hat{\varphi}$ and such that for all $x \in X_0$, $g \in G$

$$\varphi(g \cdot x) = \pi_g(\varphi(x)).$$

Proof. Let $A \subset G \times X$ be the set of (g, x) with $\hat{\varphi}(g \cdot x) = \pi_g(\hat{\varphi}(x))$. We have assumed that A is conull, and then in particular we have that for any $h \in G$

the set of (g, x) with $(g, h \cdot x) \in A$ is again conull. Then by Fubini the set

$$X_0 = \{x \in X : \text{a.e. } g, h \in G((g, h \cdot x) \in A)\}$$

is again conull; by right invariance X_0 is G-invariant.

Given any $x \in X_0$ we choose some $h \in G$ such that for almost every $g \in G$ we have $(g, h \cdot x) \in A$ and let $\varphi(x) = \pi_h^{-1}(\hat{\varphi}(h \cdot x))$; this definition does not depend on the choice of h, since given a competing choice h' we may find g_1, g_2 such that $g_1 h = g_2 h'$ and $(g_1, h \cdot x), (g_2, h' \cdot x) \in A$, and note that

$$\pi_h^{-1}(\hat{\varphi}(h \cdot x)) = \pi_h^{-1}\pi_{g_1}^{-1}(\hat{\varphi}(g_1 h \cdot x)) = \pi_{h'}^{-1}\pi_{g_2}^{-1}(\hat{\varphi}(g_2 h' \cdot x)) = \pi_{h'}^{-1}(\hat{\varphi}(h' \cdot x)).$$

A similar calculation gives that φ is equivariant on all of X_0. ∎

A parallel and well-known argument gives that a measurable action of a lcsc group G on a standard Borel probability space (X, μ) agrees almost everywhere with a Borel action of G on X (see [31]). We will generally assume only that our actions are measurable, but in light of this well known argument it is reasonable to blur the distinction between Borel actions and merely measurable actions.

LEMMA 2.2: *Mild mixing implies $\mathcal{M}$-ergodicity for $\mathcal{M}$ the class of all separable Banach spaces.*

Proof. Given $\pi : G \to \mathrm{Isom}(E)$ and equivariant $f : X \to E$, we obtain a G-invariant function

$$X \times X \to \mathbb{R}$$

$$(x, y) \mapsto \|f(x) - f(y)\|.$$

If the action of G on $X \times X$ is ergodic, which is the case assuming the original action was mild mixing, then this function must be constant almost everywhere, which would imply it is zero almost everywhere. ∎

The converse to this lemma fails. There is an action of $\mathbb{F}_2$ which is $\mathcal{M}$-ergodic for $\mathcal{M}$ the class of all separable Banach spaces but is not mildly mixing. This counterexample uses extraneous ideas, and so we put it off until the end of §3.

Nevertheless, $\mathcal{M}$-ergodicity does imply the analog of mild mixing if we restrict to taking products with *measure preserving actions.* Here we provide a definition.

Definition: A properly ergodic, measurable, non-singular action of a lcsc group G on (X, μ) is **moderately mixing** if whenever G acts in a properly ergodic, measurable, measure preserving manner on a standard Borel probability space (X', μ') the diagonal action of G on $(X \times X', \mu \times \mu')$ is again ergodic.

Lemma 2.3: *Let G be a lcsc group acting in a measurable, non-singular, ergodic manner on a standard Borel probability space (X, μ). The action of G on (X, μ) is moderately mixing if and only if it is $\{\ell^2\}$-ergodic.*

Proof. $(\Leftarrow)$: Suppose G acts by measure preserving transformations on a standard Borel probability space (Y, ν). For a contradiction assume $A \subset X \times Y$ is measurable, invariant, and neither null nor conull. At each $x \in X$ we let $A_x = \{y \in Y : (x, y) \in A\}$. Define

$$\pi : X \to L^2(Y)$$
$$x \mapsto \chi_{A_x},$$

the function which assigns to each x the characteristic function of the slice A_x. This is almost everywhere G-equivariant, and so the assumptions on X imply it is constant almost everywhere *as an element in* $L^2(Y)$. Thus we have some $A \subset Y$ such that $A_x = A$ for almost every x, and in particular for every $g \in G$ the measure of $A \Delta g \cdot A$ is zero. Replacing A by $\{y \in Y :$ for almost every $g \in G(g \cdot y \in A)\}$, gives a truly invariant subset of Y, with a contradiction to ergodicity.

$(\Rightarrow)$: Assume

$$\pi : G \to \mathrm{LinIsom}(\ell^2)$$
$$g \mapsto \pi_g$$

is a continuous homomorphism and we have π-equivariant measurable $\phi : X \to \ell^2$ with $\phi(g \cdot x) = \pi_g \cdot \phi(x)$ for almost every x, g. We assume π is not constant and try to construct a counterexample to moderate mixing.

Claim: There is an *irreducible* unitary representation

$$\rho : G \to \mathrm{LinIsom}(\mathcal{H})$$
$$g \mapsto \rho_g$$

and non-trivial measurable $\psi : X \to \mathcal{H}$ with $\psi(g \cdot x) = \rho_g \cdot \phi(x)$ for almost every x, g.

Proof of Claim. Following say [21] we find a standard measure space (Z, m) and write ℓ^2 as a direct integral of irreducible representations of G:

$$\ell^2 = \int_Z \mathcal{H}_z dz$$
$$\phi = \int_Z \phi^z dz.$$

Then for each $z \in Z$ we have a map

$$p_z : \ell^2 \to \mathcal{H}_z$$

which coordinate so that for any $v \in \ell^2$ we have

$$v = \int_Z p_z(v) dz$$

and for any $g \in G$

$$g \cdot v = \int_Z \phi^z_g \cdot p_z(v) dz.$$

Thus for almost every g, x we have at m-a.e. z

$$p_z(\phi_g \cdot \pi(x)) = \phi^z_g \cdot p_z(\pi(x));$$

we may assume that $p_z \circ \pi$ is non-trivial at every z. Thus we may appeal to Fubini to find a single z such that for almost every g, x $p_z(\phi_g \cdot \pi(x)) = \phi^z_g \cdot p_z(\pi(x))$ and finish with $\mathcal{H} = \mathcal{H}_z$, $\rho = \phi^z$, $\psi = p_z(\pi)$. ∎

Now there is a split in cases:

Case 1: The representation is finite dimensional.

Clearly there is a $U(\mathcal{H})$-invariant invariant probability measure ν' on the unit ball in $\mathcal{H}$. For each $x \in X$ we let (Y_x, ν_x) be the ergodic component of $\psi(x)$ in $((\mathcal{H})_1, \nu')$ under the action of G. By ergodicity of the G action on X, we obtain that this ergodic component is constant μ-a.e; let (Y, μ) be this ergodic component.

Now we obtain a non-trivial G-invariant function

$$f : X \times Y \to \mathbb{R}$$
$$(x, v) \mapsto \|\psi(x) - v\|.$$

Case 2: The representation is infinite dimensional.

Following [31, 5.2.13] we may find a standard Borel probability space (Y, ν) on which G acts measurably, measure preservingly, and ergodically, and for which the representation ρ is isomorphic to a direct summand of the usual representation of G on $L^2(Y, \nu)$; at that point we may assume $\mathcal{H}$ is a direct summand of $L^2(Y, \nu)$, and hence every $f \in \mathcal{H}$ is a function in $L^2(Y, \nu)$.

We obtain a non-trivial function with

$$f : X \times Y \to \mathbb{C}$$
$$(x, y) \mapsto (\psi(x))(y).$$

Unwinding the definitions we see that almost everywhere

$$f(g \cdot x, g \cdot y) = (\psi(g \cdot x))(g \cdot y) = g \cdot (\psi(x))(g \cdot y) = \psi(x)(g^{-1}g \cdot y) = \psi(x)(y),$$

since for any function $F \in L^2(Y, \nu)$ we have by definition that $g \cdot F(z) = F(g^{-1}z)$ any $z \in Y$. ∎

Schmidt and Walters have obtained a purely combinatorial characterization of mild mixing:

THEOREM 2.4 ([28]): *Let G act in a measurable, non-singular, properly ergodic manner on (X, μ). The action is mild mixing if and only if for all A measurable and neither null nor conull*

$$\mathrm{liminf}_{g \to \infty} \mu(A \Delta (g \cdot A)) > 0.$$

It would be interesting if there is a similar combinatorial characterization of when a properly ergodic non-singular action preserves ergodicity with respect to products with measure preserving, ergodic actions on standard Borel probability spaces.

We work toward a general result, due to Kaimanovich, and after Burger and Monod. In this section we see that any locally compact group has an $\mathcal{M}$-ergodic action, for $\mathcal{M}$ the class of separable Banach spaces. In the next section we review the amenability properties of this action.

Definition: Let μ be a probability measure on a lcsc group G that has the same measure class as a left invariant Haar measure. Consider the one sided product,

$$\prod_{n \in \mathbb{N}} (G, \mu),$$

equipped with the measure λ arising from the infinite product of μ. We equip this space with the shift action

$$T : \prod_{n\in\mathbb{N}} G \to \prod_{n\in\mathbb{N}} G$$

$$(g_1, g_2, g_3, \ldots) \mapsto (g_2, g_3, g_4, \ldots).$$

LEMMA 2.5 (Kaimanovich; [15]): *If E is a separable Banach space and*

$$F : \prod_{\mathbb{N}} G \to E$$
$$\pi : G \to \mathrm{Iso}(E)$$

are measurable with

$$\pi_{g_1} F((g_2, g_3, \ldots)) = F((g_1, g_2, \ldots)),$$

then F is constant almost everywhere.

Note that we do not yet assume π is a homomorphism, though actually it will be in the cases of interest to us. The measurability assumption on F is that pullbacks of sets in E which are open in the Banach space metric are λ measurable.

We need to define another semigroup action along with another action of the group G.

Definition: Let

$$S : \prod_{n\in\mathbb{N}} G \to \prod_{n\in\mathbb{N}} G$$
$$(g_1, g_2, g_3, \ldots) \mapsto (g_1 g_2, g_3, g_4, \ldots).$$

Note that the assumptions on μ guarantee that this semigroup action is non-singular with respect to the product measure λ.

We then let G act on $\prod_{\mathbb{N}} G$ by

$$\gamma \cdot (g_1, g_2, g_3, \ldots) = (\gamma g_1, g_2, g_3, \ldots).$$

This G action clearly commutes with S and is again non-singular by choice of μ.

Definition: Let $\Gamma(G,\mu)$ be the space of ergodic components of S; that is to say, we write $\prod_{\mathbb{N}} G$ as a measurable union of disjoint pieces,

$$\prod_{\mathbb{N}} G = \dot{\bigcup_{x\in\Gamma(G,\mu)}} Y_x,$$

where each Y_x is S-invariant and S acts ergodically on Y_x. (See [12]). We then let

$$b : \prod G \twoheadrightarrow \Gamma(G,\mu)$$

be the natural surjection, with $b(\vec{g}) = x$ if $\vec{g} \in Y_x$. We let $\nu = b^*[\lambda]$ be the push forward of λ, defined by $\nu[A] = \lambda[b^{-1}[A]]$.

Since the action of G commutes with S we obtain a well-defined action of G on $\Gamma(G,\mu)$ with

$$\gamma \cdot b(\vec{g}) = b(\gamma \cdot \vec{g}).$$

Again, this action is non-singular, since the pre-existing action on $(\prod_N G, \lambda)$ is non-singular.

Therefore we have a daunting total of four non-singular actions: The semigroup action provided by T; the semigroup action provided by S; the action on the original $\prod_{\mathbb{N}} G$ by G; the induced action of G on the quotient object $\Gamma(G,\mu)$. Ultimately we will only be interested in this final action of G, but it is the remarkable discovery of [15] that the other three actions cooperate to prove:

Theorem 2.6 (Kaimanovich, [15]): *Let $\mathcal{M}$ be the class of separable Banach spaces. Then the action of G on $\Gamma(G,\mu)$ is $\mathcal{M}$-ergodic.*

Proof. (In any reasonable mathematical sense, this proof has been previously given in [15]; however since our set up and notation is rather different, we recall it here for the reader's convenience.) Fix $E \in \mathcal{M}$ and $\pi : G \to \mathrm{Iso}(E)$ a continuous homomorphism. Let $f : \Gamma(G,\mu) \to E$ be G-equivariant and measurable in the usual sense. We then define $F = f \circ b$

$$F : \prod_{\mathbb{N}} G \to E,$$
$$\vec{g} \mapsto f(b(\vec{g})).$$

F is clearly measurable; on the other hand we obtain almost everywhere that

$$\pi_{g_1} F((g_2, g_3, \ldots)) = \pi_{g_1}(f(b((g_2, g_3, \ldots))) = f(g_1 \cdot b((g_2, g_3, \ldots))),$$

by the G-equivariance of f,

$$= f(b(g_1 \cdot (g_2, g_3, \ldots))),$$

by definition of the G-action on $\Gamma(G, \mu)$,

$$= f(b(g_1g_2, g_3, \ldots)) = f(b(g_1, g_2, g_3, \ldots)) = F((g_1, g_2, g_3, \ldots)),$$

by invariance of b under S; hence by 2.5 f is constant almost everywhere. ∎

By suitably composing any $\vec{g} \in \prod_{\mathbb{N}} G$ we can view such a sequence as providing a random walk through G. The shift action S then corresponds to discounting the choice of the first element in this walk. The space of ergodic components $\Gamma(G, \mu)$ is something like the space of random walks considered up to "direction" or "eventual behavior". We have failed to emphasize this point of view above; it comes through much more strongly in the original treatment of [15].

Kaimanovich's paper also proves a better result: The G-action on $\Gamma(G, \mu)$ is doubly ergodic. That is to say, any measurable, G-equivariant function $f : \Gamma(G, \mu) \times \Gamma(G, \mu) \to E$ is constant almost everywhere. The proof of this stronger result, for which we have no need, is similar and follows from a strengthening of 2.5.

3. Amenability

Definition: (See [31]; but note that, unlike Zimmer, we write our actions on the left.) Let G be a lcsc group acting measurably on (X, μ) and let H be a Polish group. A measurable function

$$\alpha : X \times G \to H$$

is a **cocycle** if for almost every $x \in X$ and all $g, h \in G$

$$\alpha(x, gh) = \alpha(hx, g)\alpha(x, h).$$

In the case that $H = \mathrm{Iso}(E)$, the isometry group of some separable Banach space, we obtain the **dual cocycle**

$$\alpha^* : X \times G \to \mathrm{Iso}(E^*)$$

by taking the adjoint in the usual way.

Definition: We say that the action of G on (X, μ) is **amenable** if whenever we have such a cocycle and a measurable assignment

$$x \mapsto A_x,$$
$$X \to K_c(E_1^*),$$

assigning to each point a weak* compact, convex subset of the unit ball E^* with the invariance property

$$\alpha^*(x, g)A_x = A_{gx}$$

for almost every x, all g, then we can find a measurable assignment

$$X \to E^*$$
$$x \mapsto a_x,$$

with each $a_x \in A_x$ and $\alpha^*(x, g)a_x = a_{g\cdot x}$ for almost every x, all g.

We are thinking of E as having the weak* topology, and so in particular measurability of the assignment $x \mapsto A_x$ is the requirement that it be measurable with respect to the induced Effros Borel structure: That is to say, if $U \subset E^*$ is weak* open, then the set of x with $A_x \cap U \neq \emptyset$ must be μ-measurable.

There is also a parallel definition we can give for amenability of equivalence relations. We do not give this definition here, but simply quote an alternative characterization from [7].

THEOREM 3.1 (Connes-Feldman-Weiss; [7]): *An equivalence relation E on a standard Borel probability space is* **amenable** *if and only if there is a conull subset A with $E|_A \leq_B E_0$.*

LEMMA 3.2 (Zimmer; [31]): *If S is an amenable G-space and X is any standard Borel probability space on which G acts measurably and non-singularly, then $X \times S$ is an amenable G-space in the usual diagonal action.*

THEOREM 3.3 (Zimmer; [30]): *If G is lcsc, then $\Gamma(G, \mu)$, as defined in §2, is an amenable G-space.*

Putting together Lemma 3.2 and Theorem 3.3 along with Theorem 2.6 we get:

COROLLARY 3.4: *An lcsc group G admits a non-singular, measurable action on a standard Borel probability space (S, m) such that for any other standard*

Borel probability space (X,ν) on which G acts measurably and in a measure preserving, properly ergodic fashion, we have:

(i) *the diagonal action of G on $(S\times X, m\times\nu)$ is properly ergodic;*
(ii) *the diagonal action of G on $(S\times X, m\times\nu)$ is amenable.*

Without placing additional restrictions on G, this corollary would seem to be the strongest we can hope for in this direction. In particular, we cannot have a mildly mixing action of $\mathbb{F}_2$ which is amenable.

PROPOSITION 3.5 (Folklore): *Let the free group $\mathbb{F}_2$ on two generators act amenably on a standard Borel probability space (S,m) by non-singular transformations. Then the action of $\mathbb{F}_2$ on (S^3, m^3) is not properly ergodic.*

Proof. Note that $\mathbb{F}_2$ acts continuously on its boundary, $\partial\mathbb{F}_2$, and hence we have an induced homomorphism $\rho:\mathbb{F}_2\to \mathrm{Iso}C(\partial\mathbb{F}_2)$, from $\mathbb{F}_2$ to the isometry group of the continuous functions on the boundary. We then obtain an induced cocycle

$$S\times\mathbb{F}_2\to \mathrm{Iso}(C(\partial\mathbb{F}_2))$$
$$(s,\sigma)\mapsto\rho(\sigma).$$

The action is amenable, and hence for $M(\partial\mathbb{F}_2)$ the space of probability measures on $\partial\mathbb{F}_2$ (and the dual of $C(\partial\mathbb{F}_2)$) we obtain a measurable equivariant assignment

$$S\to M(\partial\mathbb{F}_2),$$
$$s\mapsto\nu_s.$$

At the level of S^3 we obtain a corresponding equivariant

$$S^3\to M(\partial\mathbb{F}_2),$$
$$(s_1,s_2,s_3)\mapsto\frac{1}{3}(\nu_{s_1}+\nu_{s_2}+\nu_{s_3}).$$

On an invariant non-null set, $A\subset S^3$, the resulting measure $\nu_{s_1}+\nu_{s_2}+\nu_{s_3}$ does not concentrate on just two points. Following, e.g., [13, §C4] (or see [2] and [4] for the primary references) we may find a measurable equivariant map from the measures on $M(\partial\mathbb{F}_2)$ concentrating on more than two points, and hence there is a measurable equivariant map

$$\varphi: A\to\mathbb{F}_2.$$

This gives us a selector $A_0\subset A$ defined by

$$A_0=\{(s_1,s_2,s_3)\in A:\varphi(s_1,s_2,s_3)=e\},$$

where e is the identity in $\mathbb{F}_2$; the existence of such a selector contradicts proper ergodicity of $\mathbb{F}_2$ on A and hence on S^3. ∎

There is one last fact in this direction we need.

Lemma 3.6: *If lcsc G acts on standard Borel probability space (X,μ) by measure preserving transformations with every point having an amenable stabilizer, then if $E_G^X \leq_B E_0$, the group G is amenable.*

Proof. From [14, 2.15] we have that the equivalence relation is amenable; and then it is known from [3] that an action with amenable stabilizers and amenable equivalence relation is amenable, and thus G is amenable by [31, 4.3.3]. ∎

4. The result

Theorem 4.1: *Let G_1, G_2 be lcsc groups acting on a standard Borel probability space (X,μ) with:*

- (0) *the actions of G_1 and G_2 commute;*
- (i) *the action of $G_1 \times G_2$ is measurable (as a function from $G_1 \times G_2 \times X$ to X);*
- (ii) *the action of $G_1 \times G_2$ is measure preserving;*
- (iii) *the orbit equivalence relation E_{G_1} is not amenable;*
- (iv) *the action of G_2 is properly ergodic on all its ergodic components.*

Then the resulting orbit equivalence relation $E^X_{G_1\times G_2}$ is non-treeable.

Proof. By going to ergodic components we may actually assume that the action of $G_1 \times G_2$ is ergodic. Assume towards a contradiction that the equivalence relation is treeable, and then appealing to Corollary 1.2 we obtain that some Borel $\theta : X \to F(2^{\mathbb{F}_2})$ witnessing $E^X_{G_1\times G_2}$ Borel reducible to the free Borel action of $\mathbb{F}_2$ on the standard Borel probability space $F(2^{\mathbb{F}_2})$. In particular, we obtain a corresponding cocycle

$$\alpha : X \times G_1 \times G_2 \to \mathbb{F}_2.$$

Following the ergodic decomposition theorem of [12], we write

$$X = \dot{\bigcup_{z\in Z}} X_z$$

as a decomposition of X into G_2-invariant ergodic pieces[3]; applying the measure disintegration theorem (see for instance [16]) we obtain a measure $\hat{\mu}$ on Z along with G_2-invariant measures $(\mu_z)_{z\in Z}$ such that $\mu = \int_Z \mu_z \hat{\mu}(z)$. The assumptions of the theorem imply that the action of G_2 on each X_z is properly ergodic. This is a critical observation for the proof — if G_2 were to act essentially transitively on its ergodic components, then it would make no real contribution to the complexity of the structure of the equivalence relation, the argument in the first claim below would fail, and in fact the theorem without this assumption would admit a counterexample.

Now appealing to 3.4 we let (S, m) be a standard Borel probability space on which G_2 acts ergodically with the diagonal action on $X \times S$ amenable and such that at each fiber X_z we have the induced action of G_2 on $X_z \times S$ still properly ergodic. Let

$$\hat{\alpha} : (X \times S) \times G_2 \to \mathbb{F}_2$$

be derived from α in the obvious way:

$$\hat{\alpha}(x, s, g) = \alpha(x, e_{G_1}, g),$$

where e_{G_1} is the identity in G_1. By the amenability of the action of G_2 on $X \times S$ we may find a measurable assignment of probability measures

$$X \times S \to M(\partial\mathbb{F}_2)$$

$$(x, s) \mapsto \nu_{x,s}$$

such that $\hat{\alpha}(x, s, g) \cdot \nu_{x,s} = \nu_{g\cdot x, g\cdot s}$ for almost every $x \in X$, $s \in S$ and all $g \in G_2$.

We let $M_{\leq 2}(\partial\mathbb{F}_2)$ be the probability measures concentrating on at most two points; we let $M_{\geq 3}(\partial\mathbb{F}_2) = M(\partial\mathbb{F}_2) \setminus M_{\leq 2}(\partial\mathbb{F}_2)$ be the measures whose support is greater than two points.

The next argument through the next couple of claim parallels [1] and is similar to ideas of Zimmer, and from there Margulis and eventually Mostow.

Claim: For almost every x, s, $\nu_{x,s} \in M_{\leq 2}(\partial\mathbb{F}_2)$.

Proof of Claim. Otherwise we can obtain some ergodic component X_z such that for almost every $x \in X_z$ and for almost every $s \in S$ we have $\nu_{x,s} \in M_{\geq 3}(\partial\mathbb{F}_2)$.

[3] Here Z is the standard Borel space arising as equivalence classes of the smooth equivalence relation $x_1 E x_2$ if and only if $\forall B \in \mathcal{B}_0$(for almost every $g \in G(g \cdot x_1 \in B) \Leftrightarrow$ for almost every $g \in G(g \cdot x_2 \in B)$), for some countable Boolean algebra $\mathcal{B}_0$ which generates the Borel sets on X.

Appealing to the existence of an equivariant Borel map from $M_{\geq 3}(\partial \mathbb{F}_2)$ to $\mathbb{F}_2$, as given discussed in the course of Proposition 3.5 (again see, for instance, Theorem C4 of [13], or [2], [4]) we may find a measurable $\hat{\alpha}$-equivariant $\varphi : X_z \times S \to \mathbb{F}_2$.

This will provide a contradiction to *proper* ergodicity of the action. We let $(\sigma_i)_{i\in\mathbb{N}}$ enumerate $\mathbb{F}_2$ and at each $(x, s) \in X_z \times S$ we let $i(x, s)$ be the least i for which there exists a non-null collection of $g \in G_2$ with $\varphi(g \cdot x, g \cdot s) = \sigma_i$; since the measure quantifier preserves Borel sets (see for instance [16]), this function is Borel, and moreover the set

$$A_1 = \{(x, s) : \varphi(x, s) = \sigma_{i(x,s)}\}$$

will provide a Borel selector for $E_{G_2}^{X_z\times S}$; since $\hat{\alpha}(x, s, g) \cdot \theta(x) = \theta(g \cdot x)$ by the definition of the cocycle, we have

$$\theta(x_1) = \theta(x_2)$$

for all $(x_1, s_1), (x_2, s_2) \in A_1$ with $(x_1, s_1)E_{G_2}^{X_z\times S}(x_2, s_2)$; then using say Jankov, von Neumann uniformization (see for instance [16]) we obtain a measurable function $f : X_z \times S \to F(2^{\mathbb{F}_2})$ assigning to each (x, s) the unique value $\theta(x')$ any $(x', s') \in A_1$ that is E_{G_2} equivalent to (x, s). Our action is ergodic, and f is G_2-invariant, so f is constant almost everywhere. But that means in particular that the equivalence class $[\theta(x)]_{\mathbb{F}_2}$ is constant almost everywhere, and hence, by assumption on θ, the equivalence class $[x]_{G_2}$ is constant almost everywhere on X_z. This contradicts the proper ergodicity of the action of G_2 on $X_z \times S$. ∎

By a straightforward exhaustion argument (compare [1], [18], or even [13, Lemma 6.6]) we may find a measurable, equivariant assignment

$$X \times S \to M(\partial \mathbb{F}_2)$$

$$(x, s) \mapsto \nu_{x,s}$$

which is maximal almost everywhere. In other words, if

$$(x, s) \mapsto \nu'_{x,s}$$

is any other measurable, equivariant assignment with

$$\mathrm{supp}(\nu_{x,s}) \subset \mathrm{supp}(\nu'_{x,s})$$

(where here $\mathrm{supp}(\mu)$ denotes the support of measure μ) and $\hat{\alpha}(x, s, g) \cdot \nu'_{x,s} = \nu'_{g\cdot x,g\cdot s}$ for almost every x and g, then

$$\nu'_{x,s} = \nu_{x,s}$$

almost everywhere. Since $\nu_{x,s} \in M_{\leq 2}(\partial\mathbb{F}_2)$ almost everywhere, we may further more assume that these measures assign all points in their support equal mass — which is to say, almost every $\nu_{x,s}$ equals either δ_a or $\frac{1}{2}(\delta_{a_1} + \delta_{a_2})$.

Claim: For $g \in G_1$, the assignment

$$(x, s) \mapsto \alpha(g \cdot x, g, e_{G_2})^{-1} \cdot \nu_{g \cdot x, s}$$

is again $\hat{\alpha}$-equivariant.

Proof of Claim. Since for any $h \in G_2$ we have

$$\hat{\alpha}(x, s, h) \cdot (\alpha(g \cdot x, g, e_{G_2})^{-1} \cdot \nu_{g \cdot x, s}) = \alpha(x, h, e_{G_1}) \cdot (\alpha(g \cdot x, g, e_{G_2})^{-1} \cdot \nu_{g \cdot x, s}),$$

which by the defining identity of the cocycle and the fact that the actions of G_1 and G_2 commute equals

$$\alpha(hg \cdot x, g, e_{G_2})^{-1} \cdot (\alpha(g \cdot x, h, e_{G_1}) \cdot \nu_{g \cdot x, s}) = \alpha(hg \cdot x, g, e_{G_2})^{-1} \cdot \nu_{hg \cdot x, h \cdot s},$$

by G_2-equivariance of the cocycle, which in turn by the commutativity of the G_1, G_2 actions equals

$$= \alpha(gh \cdot x, g, e_{G_2})^{-1} \cdot \nu_{gh \cdot x, h \cdot s},$$

as required. ∎

Thus by maximality of the assignment

$$(x, s) \mapsto \nu_{x,s}$$

we have

$$\alpha(g \cdot x, g, e_{G_2})^{-1} \cdot \nu_{g \cdot x, s} = \nu_{x,s}$$

almost everywhere, and hence that $(x, s) \mapsto \nu_{x,s}$ must be equivariant under the entire action of $G_1 \times G_2$. In particular, we obtain that for almost every $(x, s) \in X \times S$ and almost every $g \in G_1$ we have some $\alpha(x, g, e_{G_2}) \cdot \nu_{x,s} = \nu_{g \cdot x, s}$. Using Fubini we may find some specific $s_0 \in S$ such that for almost every $g \in G_1$, $x \in X$

$$\alpha(x, g, e_{G_2}) \cdot \nu_{x,s_0} = \nu_{g \cdot x, s_0}$$

$$\therefore \nu_{g \cdot x, s_0} E_{\mathbb{F}_2}^{M(\partial\mathbb{F}_2)} \nu_{x, s_0}.$$

Since the equivalence relation on $M(\partial\mathbb{F}_2)$ induced by the action of $\mathbb{F}_2$ is hyperfinite (see, for instance, the appendix of [13]), we may write

$$E_{\mathbb{F}_2}^{M(\partial\mathbb{F}_2)} = \bigcup_{n\in\mathbb{N}} F_n,$$

where $(F_n)_{n\in\mathbb{N}}$ is an increasing union of Borel equivalence relation with finite classes.

Following [17] we can fix a Borel $C \subset X$ meeting each $E_{G_1}^X$ equivalence class in a countable non-empty set and a Borel function

$$f : X \to C$$

with $xE_{G_1}^X f(x)$ all $x \in X$. In particular, this gives $E_{G_1}^X \leq_B E_{G_1}^X|_C$. We then use f to transfer the measure μ to a measure $\hat{\mu}$ on C with

$$\hat{\mu}(A) = \mu(f^{-1}[A]).$$

Note that $E_{G_1}^X|_C$ will be non-amenable with respect to $\hat{\mu}$ since $E_{G_1}^X$ is non-amenable with respect to μ.

We then set

$$x_1 \hat{F}_n x_2$$

if $x_1 E_{G_1}^X x_2$ and $\nu_{x_1,s_0} F_n \nu_{x_2,s_0}$. Since $x_1 E_{G_1}^X x_2$ implies $\nu_{x_1,s_0} E_{\mathbb{F}_2}^{M(\partial\mathbb{F}_2)} \nu_{x_2,s_0}$ we have

$$\bigcup_n \hat{F}_n = E_{G_1}^X.$$

Now there are two cases, both leading to the conclusion that $E_{G_1}^X$ is Borel reducible to E_0 on a co-null set, and hence a contradiction to its non-amenability by Lemma 3.6.

Case (1): There is an n and a $\hat{\mu}$ non-null set on which $\hat{F}_n$ is non-smooth.

Claim: There is a $\hat{\mu}$ non-null set of $x \in C$ for which

$$\theta[[x]_{\hat{F}_n}],$$

the pointwise image of the set $[x]_{\hat{F}_n}$ under θ, is infinite.

Proof of Claim. Otherwise, apply Lusin–Novikov (as say found in [16]) to obtain a Borel function

$$\rho : \theta[C] \to C$$

such that for every y we have

$$\theta(\rho(y)) = y.$$

Then $\rho \circ \theta[C]$ meets each $\hat{F}_n$ equivalence class in a finite non-empty set. Then since Borel equivalence relations with finite classes are smooth and since $\hat{F}_n$ is Borel reducible to $\hat{F}_n|_{\rho\circ\theta[C]}$ we obtain that $\hat{F}_n$ is smooth, with a contradiction to case assumption. ∎

Claim: There is a $\hat{\mu}$ non-null set of x's for which the corresponding measure ν_{x,s_0} has infinite stabilizer.

Proof of claim. Suppose first that ν_{x,s_0} has finite stabilizer. Then for any other ν with $\nu E_{\mathbb{F}_2} \nu_{x,s_0}$ there are only finitely many $\sigma \in \mathbb{F}_2$ with $\sigma \cdot \nu_{x,s_0} = \nu$. Thus since each F_n is finite we obtain $\{\sigma \in \mathbb{F}_2 : \sigma \cdot \nu_{x,s_0} F_n \nu_{x,s_0}\}$ finite, and hence $\theta[[x]_{\hat{F}_n}]$ is finite.

In conclusion we have thus obtained that x is outside the $\hat{\mu}$ positive set of the last claim. ∎

As noted for instance in the appendix of [13], there are only countably many measures in $M_{\leq 2}(\partial\mathbb{F}_2)$ with infinite stabilizer, and hence by ergodicity we may assume there is some fixed ν such that almost everywhere

$$\nu_{x,s_0} E_{\mathbb{F}_2}^{M_{\leq 2}(\partial\mathbb{F}_2)} \nu.$$

Then we may find a measurable assignment

$$X \to \mathbb{F}_2$$
$$x \mapsto \sigma_x$$

such that $\sigma_x \cdot \nu_{x,s_0} = \nu$ almost everywhere. Thus if we replace the reduction θ by

$$\hat{\theta} : X \to F(2^{\mathbb{F}_2}),$$
$$x \mapsto \sigma_x \cdot \theta(x)$$

then we obtain a reduction of $E_{G_1}^X$ almost everywhere into an orbit equivalence relation the induced by the stabilizer of ν, which will necessarily be cyclic abelian, and hence hyperfinite by [14], with a contradiction to non-amenability of G_1 by Lemma 3.6.

Case (2): At every n there is a conull set on which $\hat{F}_n$ is smooth.

Hence $E^X_{G_1}$ is, on a conull set, hypersmooth — that is to say, the increasing union of smooth equivalence relations. Hence by the Kechris–Louveau dichotomy theorem of [19] we have that it is either $\geq_B E_1$ or $\leq_B E_0$. The former is impossible, since [19] shows that E_1 is not Borel reducible to any Polish group action, whilst the latter gets us into the same contradiction to the non-amenability of E_{G_1}. ∎

5. Spanning trees for product groups

Definition: For X a set, $T \subset X \times X$ is said to be a **spanning tree** if it is symmetric, irreflexive, acyclic, and, moreover, any two distinct points in X are connected by some path in T.

We will only be interested in the case that X is countable. In this event, the collection of such spanning trees is a Borel subset of $\{0,1\}^{X \times X}$ via the identification of a subset of $X \times X$ with its characteristic function. The group of all permutations of X acts on the standard Borel space of spanning trees by $g \cdot T = \{(g \cdot x_1, g \cdot x_2) : (x_1, x_2) \in T\}$.

The following is a continuous analog of a lemma presented in the discrete case with credit to Russ Lyons by the manuscript [20].

Lemma 5.1: *Let Ω be a countable set, let $\mathrm{Sym}(\Omega)$ be the group of all permutations of Ω, and let $G < \mathrm{Sym}(\Omega)$ be a subgroup. Suppose:*

(a) *G is closed in the topology of pointwise convergence;*
(b) *for any $a \in \Omega$ the subgroup $G_a (=_{\mathrm{df}} \{g \in G : g \cdot a = a\})$ is compact;*
(c) *there is a G-invariant Borel probability measure on the spanning trees on Ω.*

Then there is an action of G on some standard Borel probability space Y with:

(i) *the action measurable (as a function from $G \times Y$ to Y);*
(ii) *the action measure preserving;*
(iii) *for any $y \in Y$ the stabilizer of y in G, G_y, is compact;*
(iv) *the equivalence relation E^Y_G is treeable.*

Proof. Let $\Omega_0 \subset \Omega$ be an orbit under G. I claim that we can replace Ω by Ω_0 by obtaining an invariant probability measure on the spanning trees on Ω_0. To do this, we only need to find a Borel and invariant assignment of a spanning tree on Ω_0 to spanning trees on Ω, and this follows by the kind of argument one sees in [14]: Given a spanning tree T on Ω, we assign to each $y \in \Omega$ the point $p_T(y)$ arising as the place of first entrance into Ω_0 under a T-path from y to Ω_0; we can do this so if the path from y to $p_T(y)$ passes through y' then $p_T(y) = p_T(y')$; we then define $\hat{T}$ on Ω_0 by $x_1 \hat{T} x_2$ if and only if $x_1 \neq x_2$ and there are y_1, y_2 with $p_T(y_1) = x_1$, $p_T(y_2) = x_2$, and $y_1 T y_2$.

Thus from now on we assume that the action of G acts transitively on Ω and we fix some specific $a \in \Omega$ and let G_a be its stabilizer in G. We use $\mathcal{T}(\Omega)$ to denote the space of spanning trees on Ω. We let μ be the G-invariant measure $\mathcal{T}(\Omega)$.

Following [29] we let (X, ν) be a standard Borel probability space on which G acts freely. We only need to show that the orbit equivalence relation arising from the action of G on $y = \mathcal{T}(\Omega) \times X$ is treeable.

Consider then the equivalence relation E_{G_a} on $\mathcal{T}(\Omega) \times X$ arising from the group G_a. The responsible group is compact and hence the equivalence relation is smooth, and thus we may find a Borel selector $A \subset \mathcal{T}(\Omega) \times X$ meeting each E_{G_a}-equivalence class in exactly one point.

We need only define the treeing on A. For $(T_1, x_1), (T_2, x_2) \in A$ set $(T_1, x_2) \mathcal{G} (T_2, x_2)$ if and only if there is $g \in G$ with

$$\begin{gathered} g \cdot T_1 = T_2, \\ g \cdot x_1 = x_2, \\ (a, g(a)) \in T_2, \quad \text{or, equivalently,} \quad (a, g^{-1}(a)) \in T_1. \end{gathered}$$

This is a Borel graphing, since the set of g will be of size at most 1, and we can appeal to the uniformization theorem for Borel sets in the plane with small sections. To check it is a treeing we need to show existence of unique paths between $((T, x), (T', x')) \in E_G^Y \cap A \times A$.

We will construct a path and then go on to verify that any other $\mathcal{G}$-path must arise by the same process and in fact be the same.

We choose the unique $h \in G$ with $h \cdot x = x'$; the assumption of E_G equivalence entails $h \cdot T = T'$. We then form a non-self intersecting path $a_0 = a, a_1, a_2, \ldots, a_n = h(a)$ from a to $h(a)$ inside the tree T. We consider the set of all g_1 with $g_1^{-1}(a_0) = a_1$; this exactly determines g_1 up to a left coset of G_a, so we can choose some such g_1 with $g_1 \cdot (T, x) = (T_1, x_1) \in A$, and observe that g_1 witnesses $(T, x) \mathcal{G} (T_1, x_1)$. Applying the same recipe we may choose some g_2

with $g_2^{-1}(a_0) = g_1(a_2)$ and $(T_2, x_2) = g_2 \cdot (T_1, x_1) \in A$; since $g_1^{-1}(a_0) = a_1$

$$g_1(a_1) = a_0,$$

and from and $(a_1, a_2) \in T$ we obtain

$$(g_1(a_1), g_1(a_2)) \in T_1 = g_1 \cdot T$$

$$\therefore (a_0, g_1(a_2)) = (a_0, g_2^{-1}(a_0)) \in T_1$$

$$\therefore (T_1, x_1)\mathcal{G}(T_2, x_2).$$

We continue successively, obtaining at each $i < n$ the pair (T_i, x_i) along with $g_1, g_2, \dots g_{i-1}$

$$(T_i, x_i) = g_i g_{i-2} \cdots g_1 \cdot (T, x),$$

$$g_i g_{i-2} \cdots g_1(a_i) = a_0,$$

and hence

$$(a_0, g_i g_{i-1} \cdots g_1(a_{i+1})) = (g_i g_{i-1} \cdots g_1(a_i), g_i g_{i-1} \dots g_1(a_{i+1})) \\ \in g_{i-1} g_{i-2} \cdots g_1 \cdot T = T_i.$$

We then choose g_{i+1} subject to the requirement that

$$g_{i+1}^{-1}(a_0) = g_i g_{i-1} \cdots g_1(a_{i+1})$$

and $(T_{i+1}, x_{i+1}) =_{\text{df}} g_{i+1} \cdot (T_i, x_i) \in A$. This keeps the happy game playing for another round, since $g_{i+1} g_i \cdots g_1(a_{i+1}) = a_0$ and $(T_{i+1}, x_{i+1})\mathcal{G}(T_i, x_i)$. Eventually we end up with $g = g_n g_{n-1} \cdots g_1$ with the properties that $g \cdot (T, x) = (T_n, x_n) \in A$ and $g^{-1} \cdot a_0 = a_n$, which in particular entails $G_a g = G_a h$; then the assumption that A selects exactly one point from each coset will entail $(T_n, x_n) = (T', x')$ as required.

Alternatively, given any other $\mathcal{G}$ path,

$$(S_0, y_0) = (T, x), (S_1, y_1), \dots, (S_m, y_m) = (T', x')$$

we reverse the engineering above and find $h_1, h_2, \dots, h_m$ with each $h_i \cdot y_{i-1} = y_i$ and then let $b_i = h_1^{-1} h_2^{-1} \cdots h_i^{-1}(a_0)$. The assumption of $\mathcal{G}$-adjacency implies each

$$(h_{i+1}^{-1}(a_0), a_0) \in S_i = h_i h_{i-1} \cdots h_1 \cdot T$$

$$\therefore (h_1^{-1} h_2^{-1} \cdots h_i^{-1} h_{i+1}^{-1}(a_0), h_1^{-1} h_2^{-1} \cdots h_i^{-1}(a_0) \in T)$$

$$\therefore (b_{i+1}, b_i) \in T.$$

Note, however, that we must also end up with $h_m h_{m-1} \cdots h_1 = h$, and hence $b_m = a_n$, and in particular $a, b_1, b_2, \ldots, b_m = a_n$ traces out a path in T from a to a_n. It then follows from T being a tree and the assumptions on our set A that either that either $m = n$ and each $a_i = b_i$ or there is a repeated vertex – $(S_i, y_i) = (S_j, y_j)$ some $i \neq j \leq n$ – along the way. ∎

This lemma granted, Theorem 0.4 follows from Theorem 4.1. In the situation they describe we have a product of lcsc automorphism groups, the first of which is assumed to be non-amenable and the second of which is implicitly assumed to be non-compact. Their free action on any probability space must be non-treeable by Theorem 4.1, and hence by the lemma above there can be no invariant probability measure on the spanning trees of $X \times Y$ in the statement of Theorem 0.4.

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